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CONFORMAL MAPPING OF A FAMILY OF REAL CONICS UPON ANOTHER.[*]

By T. H. Gronwall.

1. **Introduction.** Let $z = x + yi$ and $w = u + vi$ be two complex variables, and $w(z)$ an analytic function of z; then the relation

$$w = w(z) \tag{1}$$

defines a conformal map of the z-plane upon the w-plane. It is the purpose of the present paper to determine *all* functions $w(z)$ such that there exists a family (containing at least one real parameter) of *real* conics in the z-plane which is transformed by (1) into a family (obviously with an equal number of parameters) of real conics in the w-plane. A summary of the results is given in the last paragraph. The particular case where the conics in the w-plane are straight lines parallel to the real axis has been investigated by Von der Mühll[†] and Meyer.[‡] The fact that x and y appear in (1) in the combination $x + yi$ immediately suggests the use of the isometric coördinates

$$z = x + yi, \qquad \bar{z} = x - yi.$$

We have conversely

$$x = \frac{1}{2}(z + \bar{z}), \qquad y = \frac{1}{2i}(z - \bar{z}),$$

and

$$x^2 + y^2 = z\bar{z} = |z|^2.$$

2. **Equations of the straight line and circle in isometric coördinates.** The equation of a straight line through $x = p$ ($\geqq 0$), $y = 0$ and perpendicular to the real axis is $x = p$ or $z + \bar{z} - 2p = 0$; replacing z by $ze^{-\alpha i}$ we obtain the equation of a straight line, the perpendicular on which from the origin is of length p and forms an angle α with the real axis:

$$e^{-\alpha i}z + e^{\alpha i}\bar{z} - 2p = 0. \tag{2}$$

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[*] Read before the American Mathematical Society,

[†] K. Von der Mühll, "Ueber die Abbildung von Ebenen auf Ebenen," Journ. f. Math., vol. 69 (1868), pp. 264–285.

[‡] Hans Meyer, "Ueber die von geraden Linien und von Kegelschnitten gebildeten Isothermen, sowie ueber einige von speciellen Curven dritter Ordnung gebildeten Isothermen," Diss. Göttingen, 1879 (Zürich, Zürcher und Furrer).

The equation of a circle with center at z_1 and radius r is $|z - z_1| = r$ or $(z - z_1)(\bar{z} - \bar{z}_1) = r^2$ or finally

$$zz̄ - \bar{z}_1 z - z_1 \bar{z} + z_1 \bar{z}_1 - r^2 = 0. \tag{3}$$

Consequently, the equation of a straight line or a circle may be written in the form

$$\bar{z} = \frac{az + b}{cz + d}, \tag{4}$$

where

$$ad - bc = 1. \tag{5}$$

When (4) is to be identical with (2), we find $a = -ke^{-\alpha i}$, $b = 2kp$, $c = 0$, $d = ke^{\alpha i}$, the factor of proportionality being determined by (5), which gives $k^2 = -1$. Consequently

$$\begin{aligned} a &= -\epsilon i e^{-\alpha i}, \qquad b = 2\epsilon p i, \\ c &= 0, \qquad d = \epsilon i e^{\alpha i}, \qquad \epsilon = \pm 1, \end{aligned} \tag{6}$$

and the conditions that these equations shall be satisfied by real values of p and α, i.e., that (4) shall represent a real straight line, are

$$\bar{a} = d, \qquad b + \bar{b} = 0, \qquad c = 0. \tag{7}$$

In the same way it is seen that when (4) represents a real circle with center z_1 and radius r, we must have

$$\bar{a} = d, \qquad b + \bar{b} = 0, \qquad c + \bar{c} = 0, \qquad c \neq 0, \tag{8}$$

$$a = \epsilon i \frac{\bar{z}_1}{r}, \qquad b = \epsilon i\left(r - \frac{z_1 \bar{z}_1}{r}\right), \qquad c = \frac{\epsilon i}{r}, \qquad d = -\epsilon i \frac{z_1}{r}, \qquad \epsilon = \pm 1. \tag{9}$$

3. **Equations of the other conics in isometric coördinates.** An ellipse (or hyperbola) with foci at 1 and -1 and major (or transverse) axis $2a$ is defined by

$$|z - 1| \pm |z + 1| = 2a. \tag{10}$$

Transposing and squaring, we obtain

$$(z + 1)(\bar{z} + 1) = 4a^2 + (z - 1)(\bar{z} - 1) - 4a|z - 1|$$

or

$$2a|z - 1| = 2a^2 - z - \bar{z};$$

squaring again and solving the resulting quadratic for $\bar{z}$, we find

$$\bar{z} = (2a^2 - 1)z \pm \sqrt{(4a^4 - 4a^2)(z^2 - 1)},$$

or writing $2a^2 - 1 = k$ and omitting the double sign before the square root,

$$\bar{z} = kz + \sqrt{(k^2 - 1)(z^2 - 1)}. \tag{11}$$

From the definition of k, it follows that $k > -1$, and since $a > 1$ when (10) represents an ellipse, it is seen that (11) represents an ellipse when $k > 1$ but an hyperbola when $1 > k > -1$. When $k = 0$, the hyperbola is equilateral.

Replacing z by

$$\frac{2z - z_1 - z_2}{z_1 - z_2}$$

in (11) we find as the equation of an ellipse or hyperbola with foci at z_1 and z_2

$$\bar{z} = k\frac{\bar{z}_1 - \bar{z}_2}{z_1 - z_2} z + \frac{\bar{z}_1 + \bar{z}_2}{2} - \frac{k}{2}\frac{\bar{z}_1 - \bar{z}_2}{z_1 - z_2}(z_1 + z_2) \tag{12}$$
$$+ \frac{\bar{z}_1 - \bar{z}_2}{z_1 - z_2}\sqrt{(k^2 - 1)(z - z_1)(z - z_2)}.$$

The equation of a parabola with the origin as focus, the real axis as axis and the distance $\frac{1}{2}p$ from focus to directrix is

$$|z| = \tfrac{1}{2}(z + \bar{z} + p)$$

or squaring and solving for $\bar{z}$,

$$\bar{z} = z - p + 2i\sqrt{pz}. \tag{13}$$

Replacing z by $(z - z_1)e^{-\alpha i}$, we find the equation of a parabola with focus at z_1 and its axis forming an angle α with the real axis:

$$\bar{z} = e^{-2\alpha i}z + \bar{z}_1 - z_1 e^{-2\alpha i} - pe^{-\alpha i} + 2i\sqrt{pe^{-3\alpha i}(z - z_1)}. \tag{14}$$

From (12) and (14) it is seen that the equation of any conic which is not a circle or a straight line, may be written in the form

$$\bar{z} = az + b + \sqrt{cz^2 + 2dz + e}, \tag{15}$$

and that its foci are the zeros of the expression under the radical sign.

4. **Condition that two given one-parameter families of real analytic curves shall correspond in a conformal map.** Let $f(z, t)$ be an analytic function of z and t; for real values of t, the equation

(16)

defines a one-parameter family of analytic curves. At a point $z = z_0$, $t = t_0$ where t_0 is real, $\partial f/\partial z \neq 0$ and $f(z_0, t_0) = \bar{z}_0$, the preceding equation takes the form

$$\bar{z} - \bar{z}_0 = \sum_{m+n>0} (a_{mn} + ib_{mn})(z - z_0)^m (t - t_0)^n, \tag{17}$$

where all a and b are real, and $a_{10} + ib_{10} \neq 0$. By Weierstrass' preparation theorem, this equation may be solved for $z - z_0$, giving as result a power series in $\bar{z} - \bar{z}_0$ and $t - t_0$, and in order that (16) shall represent a family of real curves, it is necessary and sufficient that the solution of (17) in respect to $z - z_0$ shall be identical with the result of changing all complex quantities into their conjugates in (17), or

$$z - z_0 = \sum_{m+n>0} (a_{mn} - ib_{mn})(\bar{z} - \bar{z}_0)^m(t - t_0)^n.$$

Denoting by f or φ a power series in one or more variables, and by $\bar{f}$ or $\bar{\varphi}$ the result of changing its coefficients only into their conjugates, the necessary and sufficient condition that (16) represent a family of real curves is therefore that the equation

$$z = \bar{f}(\bar{z}, t) \tag{18}$$

shall reduce to an identity in z and t upon substitution of the value of $\bar{z}$ given by (16).

Now let

$$\bar{w} = F(w, t) \tag{19}$$

define a family of real analytic curves in the w-plane, and suppose that the analytic function

$$w = w(z) \tag{20}$$

maps the z-plane upon the w-plane in such a manner that the curves (16) are transformed into the curves (19). A necessary condition is found by differentiating (20):

$$dw = w'dz,$$

where $w' = dw/dz$, and taking conjugates, whence

$$d\bar{w} = \bar{w}' \cdot d\bar{z}.$$

By means of (16), (19) and (20), the last equation becomes

$$\frac{\partial F}{\partial w}\frac{dw}{dz}dz + \frac{\partial F}{\partial t}dt = \bar{w}'\left(\frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial t}dt\right);$$

equating coefficients of dz and dt on both sides, and eliminating $\bar{w}'$, we obtain

$$\frac{\dfrac{\partial F}{\partial w}\dfrac{dw}{dz}}{\dfrac{\partial F}{\partial t}} = \frac{\dfrac{\partial f}{\partial z}}{\dfrac{\partial f}{\partial t}}. \tag{21}$$

This condition is also sufficient, for if (20) is an analytic function independent of t which satisfies (21) the equation $d\bar{w} = \bar{w}'d\bar{z}$ becomes by (16) and (21)

$$d\bar{w} = \frac{\bar{w}' \dfrac{\partial f}{\partial t}}{\dfrac{\partial F}{\partial t}} \, dF(w, t),$$

so that $\bar{w}$ is a function of $F(w, t)$ alone, or

$$F(w, t) = \varphi(\bar{w}). \tag{22}$$

Taking conjugates, and observing that $\bar{F}(\bar{w}, t) = w$ since (19) defines a family of real curves, we find $w = \bar{\varphi}(w)$, and consequently $\bar{w} = \varphi(\bar{w})$, so that the curves (22) into which (16) are transformed by (20), are identical with the given curves (19). Writing (21) for two different values of t, say t_0 and t, we may eliminate dw/dz and obtain

$$\frac{\dfrac{\partial F}{\partial w}}{\dfrac{\partial F}{\partial t}} : \left(\frac{\dfrac{\partial F}{\partial w}}{\dfrac{\partial F}{\partial t}} \right)_{t=t_0} = \frac{\dfrac{\partial f}{\partial z}}{\dfrac{\partial f}{\partial t}} : \left(\frac{\dfrac{\partial f}{\partial z}}{\dfrac{\partial f}{\partial t}} \right)_{t=t_0} . \tag{23}$$

If this is an identity in w, z and t, each member must be a function of t alone, say $\psi(t)$, and we have

$$\frac{\dfrac{\partial f}{\partial z}}{\dfrac{\partial f}{\partial t}} = \varphi(z)\psi(t), \qquad \frac{\dfrac{\partial F}{\partial w}}{\dfrac{\partial F}{\partial t}} = \Phi(w)\psi(t), \qquad \Phi(w)\frac{dw}{dz} = \varphi(z). \tag{24}$$

In this case, the z- and w-planes may be mapped on an auxiliary Z-plane in such a manner that the curves (16) and (19) correspond to a family of parallel straight lines in the Z-plane. For define the map of the z-plane on the Z-plane by

$$\frac{dZ}{dz} = \varphi(z); \tag{25}$$

the curves (16) are then evidently mapped into a family of real curves $\bar{Z} = G(Z, t)$ in the Z-plane, and we have the equation, corresponding to (21),

$$\frac{\dfrac{\partial G}{\partial Z}}{\dfrac{\partial G}{\partial t}} \frac{dZ}{dz} = \frac{\dfrac{\partial f}{\partial z}}{\dfrac{\partial f}{\partial t}}. \tag{26}$$

or using (25) and the first of (24),

$$\frac{1}{\psi(t)} \frac{\partial G}{\partial Z} - \frac{\partial G}{\partial t} = 0. \tag{27}$$

The left member being the jacobian of G and $Z + \int[dt/\psi(t)]$, the latter expression is a function $\chi(G)$ of G, so that the equation of the curves in the Z-plane corresponding to (16) is

$$\chi(\bar{Z}) = Z + \int \frac{dt}{\chi(t)};$$

changing to conjugates, we have

$$\bar{Z} + \int \frac{dt}{\bar{\psi}(t)} = \bar{\chi}(Z)$$

so that, since $\bar{Z} = G(Z, t)$,

$$G(Z, t) = \bar{\chi}(Z) - \int \frac{dt}{\bar{\psi}(t)}.$$

Substituting in (27), we find

$$\bar{\chi}'(Z) = -\frac{\psi(t)}{\bar{\psi}(t)} = \text{constant} = a,$$

and ψ and $\bar{\psi}$ being conjugates, we have $|a| = 1$ or $a = -e^{-2\alpha i}$. The curves $\bar{Z} = G(Z, t)$ are consequently the parallel straight lines

$$e^{-\alpha i}Z + e^{\alpha i}\bar{Z} + \int \frac{e^{-\alpha i}}{\psi(t)}\,dt = 0.$$

Conversely, it follows from (26) that when the curves $\bar{Z} = G(Z, t)$ are parallel straight lines, the quotient $(\partial f/\partial z) : (\partial f/\partial t)$ must be the product of a function of z by a function of t. We remark finally that when (23) is not an identity in w, z and t, this equation gives w as a function of z without integration, and if (16) and (19) are both algebraic curves, $w(z)$ is an algebraic function of z.

5. **Cases where the conics in both the z- and the w-plane are all circles or straight lines.** When a one-parameter family of circles or straight lines in the z-plane

$$(28) \qquad \bar{z} = \frac{a(t)z + b(t)}{c(t)z + d(t)}, \qquad ad - bc = 1$$

is mapped into the circles or straight lines in the w-plane

$$(29) \qquad \bar{w} = \frac{A(t)w + B(t)}{C(t)w + D(t)}, \qquad AD - BC = 1,$$

the equation (21) takes the form

$$(30) \qquad (lz^2 + mz + n)\frac{dw}{dz} = Lw^2 + Mw + N,$$

where

$$
\begin{aligned}
l &= a'(t)c(t) - c'(t)a(t),\\
m &= a'(t)d(t) - d'(t)a(t) + b'(t)c(t) - c'(t)b(t),\\
n &= b'(t)d(t) - d'(t)b(t)
\end{aligned}
\tag{31}
$$

with corresponding expressions for L, M, N in terms of A, B, C, D. Giving t a constant value t_0 in (30), this equation takes the form

$$\frac{dw}{L_0w^2 + M_0w + N_0} = \frac{dz}{l_0z^2 + m_0z + n_0};$$

using the well-known property of a fractional linear substitution on z to transform any circle into a circle, we may reduce the last equation to a simpler form. Denoting the zeros of $l_0z^2 + m_0z + n_0$ by z_1 and z_2 (or the single finite zero by z_1 when $l_0 = 0$), we have

$$
\begin{aligned}
\frac{dz}{l_0z^2 + m_0z + n_0} &= \frac{1}{l_0(z_1 - z_2)} \cdot \frac{z - z_2}{z - z_1} d\left(\frac{z - z_1}{z - z_2}\right) &&\text{when} \quad z_1 \neq z_2,\\
&= d\left(\frac{1}{l_0(z_1 - z)}\right) &&\text{"} \quad z_1 = z_2,\\
&= \frac{1}{m_0} \cdot \frac{1}{z - z_1} d(z - z_1) &&\text{"} \quad l_0 = 0,\ m_0 \neq 0,\\
&= d\left(\frac{1}{n_0}z\right) &&\text{"} \quad l_0 = m_0 = 0,
\end{aligned}
$$

and making the linear substitution indicated by the form of the expressions to the right, as well as the corresponding substitutions on w, our equation may be reduced to one of the following four forms, k being constant:

$$
\begin{aligned}
\frac{dw}{w} &= k\frac{dz}{z},\\
dw &= \frac{dz}{z},\\
\frac{dw}{w} &= dz,\\
dw &= dz.
\end{aligned}
\tag{32}
$$

Consider first the case (32_1). Substituting the value of $dw\ dz$ from this equation in (30) the latter becomes

$$Lzw^2 + [- klz^2 + (M - km)z - kn]w + Nz = 0;$$

differentiating, and substituting the value of $du\ dz$ from (32_1), we find

$(2k+1)Lzw^2 + [-k(k+2)lz^2 + (k+1)(M-km)z - k^2n]w + Nz = 0,$ and subtracting and dividing by kw,

$$2Lzw - (k+1)lz^2 + (M-km)z - (k-1)n = 0.$$

Differentiating, substituting the value of dw/dz, and eliminating w, we obtain

$$(k^2-1)lz^2 - k(M-km)z + (k^2-1)n = 0.$$

When $k = \pm 1$, (32_1) gives $w = \text{const. } z$ and $w = \text{const.}/z$ respectively, and these linear substitutions transform every circle into a circle. When $k^2 \neq 1$, we have

$$l = 0, \qquad M = km, \qquad n = 0,$$

the preceding equations give

$$L = 0, \qquad N = 0,$$

and (30) reduces to (32_1).

In discussing the equations $l = n = 0$, we distinguish two cases. First, let $a \neq 0$; then, by (31), $l = 0$ gives $c = c_1 a$, and $n = 0$, $b = c_2 d$, where c_1 and c_2 are constants. By the conditions of reality (7) or (8), we have $d = \bar{a}$ so that $b = c_2\bar{a}$, hence $a =$ constant would make b, c and d constants, and our family of circles would reduce to a single circle. Equation (5) now gives $a\bar{a}(1 - c_1c_2) = 1$, and $c + \bar{c} = 0$, $c_1 a + \bar{c}_1\bar{a} = 0$, or multiplying by a and using the preceding equation, $c_1(1 - c_1c_2)a^2 + c_2 = 0$; since a is not a constant, we must have $c_1 = c_2 = 0$ or $b = c = 0$. From (6) it now follows that

$$a = -\epsilon i e^{-\theta i}, \qquad d = \epsilon i e^{\theta i}, \qquad p = 0,$$

so that (28) represents all straight lines through the origin, and (31_2) gives

$$m = -2i\frac{d\theta}{dt}.$$

The second case, $a = 0$, gives $d = \bar{a} = 0$, and consequently $l = n = 0$. Since now $c \neq 0$ by (5), it follows from (9) that

$$b = \epsilon i r, \qquad c = \frac{\epsilon i}{r}, \qquad z_1 = 0,$$

so that (28) represents all circles with the origin as center, and by (31)

$$m = -2\frac{dr}{dt}.$$

Similarly, the equations $L = N = 0$ give rise to the straight lines through the origin with

$$A = -\epsilon i e^{-\vartheta i}, \qquad D = \epsilon i e^{\vartheta i}, \qquad M = -2i\frac{d\vartheta}{dt},$$

and the circles with the origin as center with

$$B = \epsilon i R, \qquad C = \frac{\epsilon i}{R}, \qquad M = -2\frac{dR}{dt}.$$

Since θ, ϑ, r and R are real, the equation $M = km$ now shows that the constant k is either real or purely imaginary. In the former case, the straight lines through the origin correspond in the two planes, and so do the concentric circles. When k is purely imaginary, the straight lines in one plane correspond to the circles in the other.

Proceeding to the case (32_2), we obtain from (30)

$$Lw^2 + Mw + N - lz - m - \frac{n}{z} = 0.$$

Differentiating, using (32_2) and multiplying by z, we find

$$2Lw + M - lz + \frac{n}{z} = 0,$$

and repeating this process,

$$2L - lz - \frac{n}{z} = 0,$$

whence $l = n = 0$. Hence, as in the preceding case, the curves (28) are the straight lines through the origin and the circles with the origin as center. On a line through the origin, we have $\bar{z}/z =$ constant, and hence, by (32_2),

$$d\bar{w} - dw = \frac{d\bar{z}}{\bar{z}} - \frac{dz}{z} = 0,$$

or $\bar{w} - w =$ constant. Similarly, for a circle with the origin as center, $z\bar{z} =$ const., and by (32_2), $\bar{w} + w =$ const. Consequently, in the case (32_2), $w = \log z +$ const. maps the straight lines through the origin in the z-plane into the lines parallel to the real axis in the w-plane and the circles with the origin as center in the z-plane into the straight lines parallel to the imaginary axis in the w-plane. The case (32_3) reduces to the preceding upon replacing z by w, and (32_4) obviously transforms all circles or straight lines into circles or straight lines.

6. **Cases where the conics in one plane are neither circles nor straight lines.** Permuting w and z if necessary, we may assume that the conics in question lie in the z-plane and have the equation (15) with coefficients depending on t, or

$$\bar{z} = a(t)z + b(t) + \sqrt{c(t)z^2 + 2d(t)z + e(t)}. \tag{33}$$

When the corresponding conics in the w-plane are the straight lines or circles (29), equation (21) becomes

$$\begin{aligned}(34)\quad &\frac{1}{Lw^2 + Mw + N}\frac{dw}{dz}\\ &\qquad = \frac{a\sqrt{cz^2 + 2dz + e} + cz + d}{(a'z + b')\sqrt{cz^2 + 2dz + e} + \frac{1}{2}(c'z^2 + 2d'z + e')},\end{aligned}$$

where $a' = da(t)/dt$ etc., and L, M and N are defined by (31).

When the conics in the w-plane are

$$(35)\qquad \bar{w} = A(t)w + B(t) + \sqrt{C(t)w^2 + 2D(t)w + E(t)},$$

equation (21) becomes

$$\begin{aligned}(36)\quad &\frac{A\sqrt{Cw^2 + 2Dw + E} + Cw + D}{(A'w + B')\sqrt{Cw^2 + 2Dw + E} + \frac{1}{2}(C'w^2 + 2D'w + E')}\frac{dw}{dz}\\ &\qquad = \frac{a\sqrt{cz^2 + 2dz + e} + cz + d}{(a'z + b')\sqrt{cz^2 + 2dz + e} + \frac{1}{2}(c'z^2 + 2d'z + e')}.\end{aligned}$$

The foci of the conics in the z-plane being the branch points of the radical in the right-hand members of (34) and (36), we shall distinguish the case when one at least of these foci is variable with t from the case where the foci are fixed, and the former case will be subdivided according as the variable foci are or are not branch points of the right-hand members in (34) and (36).

7. **Cases where the conics in the z-plane have a focus variable with t but which is not a branch point of the right-hand member in (34) and (36).** When z turns around the focus in question, the radical in (33) changes its sign, and by hypothesis, the right-hand member in (34) or (36) remains unchanged. Being a linear fraction in the radical, this right-hand member must consequently be independent of it, that is

$$\tfrac{1}{2}a(c'z^2 + 2d'z + e') - (a'z + b')(cz + d) = 0,$$

whence

$$(37)\qquad \begin{aligned} ac' - 2a'c &= 0,\\ ad' - a'd - b'c &= 0,\\ ae' - 2b'd &= 0.\end{aligned}$$

First, assume that $a(t)$ is not identically zero. The differential equations (37) are then readily integrated and give

$$(38)\qquad \begin{aligned} c &= c_1a^2,\\ \frac{d}{a} &= c_1b + c_2,\\ e &= c_1b^2 + 2c_2b + c_3,\end{aligned}$$

where c_1, c_2, c_3 are constants. Assuming our conics to be the central conics (12), where k, z_1 and z_2 now depend on t, (38_1) gives

$$\frac{k^2 - 1}{k^2} = c_1,$$

so that k is a constant; (38_2) now becomes

$$\frac{k^2 - 1}{2k}(\bar{z}_1 + \bar{z}_2) + c_2 = 0,$$

so that $z_1 + z_2$ is a constant, which may be made equal to zero by a change of origin. Hence $z_2 = -z_1$, $c_2 = 0$ and $b = 0$, (38_3) gives $(1 - k^2)z_1^2 = c_3$, so that z_1 is a constant, and we have a single conic instead of a family. When our conics are the parabolas (14), we have $c = 0$ whence $c_1 = 0$ by (38_1), and (38_2) becomes

$$-2pe^{-\alpha i} = c_2,$$

so that both p and α are constants. By a rotation of the axes, we may make $\alpha = 0$, and (38_3) becomes $4p(p - \bar{z}_1) + c_3 = 0$, so that z_1 is a constant, and we have again a single conic instead of a family. There remains the case where $a(t) = 0$, and since the coefficient of z in (14) is different from zero, we have only to consider (12) with $k = 0$, representing a family of equilateral hyperbolas. Since $c \neq 0$, (37_2) gives $b' = 0$ or $\bar{z}_1 + \bar{z}_2 =$ const., and (37_3) is satisfied; moving the origin, we may make the constant equal to zero, so that $z_2 = -z_1$, and (12) becomes

$$\bar{z} = \frac{\bar{z}_1}{z_1}\sqrt{z_1^2 - z^2}. \tag{39}$$

Writing

$$z_1 = re^{\theta i}, \tag{40}$$

we have

$$c = -e^{-4\theta i}, \qquad d = 0, \qquad e = r^2e^{-2\theta i}, \tag{41}$$

and the right-hand member of (34) and (36) being independent of the radical, it is equal to

$$\frac{2cz}{c'z^2 + e'}. \tag{42}$$

Consider first the case where the z-plane may be mapped on an auxiliary Z-plane, the conics in the z-plane corresponding to parallel straight lines in the Z-plane. By § 4, this is the case when (42) becomes the product of a function of z by a function of t, which is possible in two ways: $c' = 0$ or $c' \neq 0$ and e'/c' constant. Assuming $c' = 0$, it follows from (41) that θ is a constant (that is, a parameter independent of t). By (25) and (42), we obtain $dZ/dz = 2z$, so that making the integration constant zero by moving the origin of the Z-plane

$$Z = z^2, \tag{43}$$

and by (39) and (41), it is seen that the family of all equilateral hyperbolas with the origin as center (having two parameters r and θ)

$$e^{-2\theta i}z^2 + e^{2\theta i}\bar{z}^2 = r^2 \tag{44}$$

is transformed by (43) into the family of all straight lines in the Z-plane

$$e^{-2\theta i}Z + e^{2\theta i}\bar{Z} = r^2. \tag{45}$$

In particular, giving θ a constant value, (44) becomes the family of all equilateral hyperbolas with the origin as center and their foci on the straight line $z_1 = re^{\theta i}$, while (45) represents parallel straight lines. For those lines (45) which pass through a fixed point $Z_0 = a^2e^{2\alpha i}$ we have

$$r^2 = e^{-2\theta i}Z_0 + e^{2\theta i}\bar{Z}_0 = 2a^2 \cos 2(\theta - \alpha),$$

and the corresponding equilateral hyperbolas in the z-plane have their foci at

$$\pm z_1 = \pm a\sqrt{2 \cos 2(\theta - \alpha)} \cdot e^{\theta i}. \tag{46}$$

The equations of the hyperbolas and straight lines in question are

$$e^{-2\theta i}(z^2 - a^2e^{2\alpha i}) + e^{2\theta i}(\bar{z}^2 - a^2e^{-2\alpha i}) = 0,$$

$$e^{-2\theta i}(Z - a^2e^{2\alpha i}) + e^{2\theta i}(\bar{Z} - a^2e^{-2\alpha i}) = 0.$$

Multiplying z and Z by suitable constants, we may make $a = 1$ and $\alpha = 0$, and it now follows from (25) and (42) that the function

$$Z = \log (z^2 - 1) \tag{47}$$

maps the one-parameter family of equilateral hyperbolas

$$e^{-2\theta i}(z^2 - 1) + e^{2\theta i}(\bar{z}^2 - 1) = 0 \tag{48}$$

with center at the origin and foci

$$\pm z_1 = \pm \sqrt{2 \cos 2\theta}e^{\theta i} \tag{49}$$

situated at the end points of the diameters of a lemniscate with its foci at ± 1, upon the straight lines

$$\bar{Z} - Z = -\left(\frac{\pi}{2} + 4\theta\right)i \tag{50}$$

parallel to the real axis in the Z-plane.

We now pass on to the cases where $c' \neq 0$ and e'/c' is not independent of t; from the remark at the end of § 4, it follows that w is an algebraic function of z. Let us assume first that the conics in the w-plane are the straight lines or circles (29); by (42), it follows that (34) becomes

$$\frac{1}{Lw^2 + Mw + N}\frac{dw}{dz} = \frac{2cz}{c'z + e'}, \tag{51}$$

and giving t a constant value, making a linear transformation on w according to (32), and observing that w must be an algebraic function of z, we obtain from (51)

$$\frac{1}{w}\frac{dw}{dz} = \frac{2mz}{z^2 + c_1^2},$$

whence

$$w = (z^2 + c_1^2)^m,$$

where m is real and rational. Replacing, if necessary, w by $1/w$, we may assume $m > 0$, and substituting in (51), we find

$$m(z^2 + c_1^2)^{m-1}(c'z^2 + e') = cL(z^2 + c_1^2)^{2m} + cM(z^2 + c_1^2)^m + cN.$$

Letting $z \to \infty$, we find $L = 0$, $mc' = cM$, so that the preceding equation becomes

$$m(z^2 + c_1^2)^{m-1}(e' - c_1^2c') = cN,$$

and since $e' - c_1^2c'$ is not identically zero by hypothesis, we must have $m = 1$, whence, moving the origin in the w-plane, so that $c_1 = 0$,

$$w = z^2.$$

Since (44) represents all equilateral hyperbolas in the z-plane with the origin as center, it follows at once that the present case reduces to that defined by (43), (44) and (45) upon replacing w by Z. In the second place, assume the conics in the w-plane to be (35); since w is an algebraic function of z, and the right-hand member in (36) is (42) and has no branch points variable with t, the same must be true of the left-hand member, i.e., this must be of the type (42) unless the foci of (35) are fixed. Consequently (36) reduces to

$$\frac{2Cw}{C'w^2 + E'}\frac{dw}{dz} = \frac{2cz}{c'z^2 + e'}.$$

Giving t a constant value and multiplying w by a suitable constant, we obtain a differential equation giving upon integration

$$w^2 + c_2^2 = (z^2 + c_1^2)^m,$$

where m is rational since $w(z)$ is algebraic. Substituting in the preceding equation, we obtain

$$(mCc' - cC')(z^2 + c_1^2)^m + mC(e' - c_1^2c')(z^2 + c_1^2)^{m-1} - c(E' - c_2^2C') = 0,$$

and if $m \neq 1$, it follows that $e' = c_1^2c'$ contrary to hypothesis. Hence $m = 1$, and writing c_1^2 instead of $c_1^2 - c_2^2$, we find

$$w^2 = z^2 + c_1^2 \tag{52}$$

where c_1 may be assumed real (multiplying z and w by a constant).

Noting that (44) represents all the equilateral hyperboles (39) under consideration, it follows at once that (52) transforms the two parameter family of equilateral hyperbolas (44) into itself with a changed parameter r:

$$e^{-2\theta i}w^2 + e^{2\theta i}\bar{w}^2 = r^2 + 2c_1^2 \cos 2\theta. \tag{53}$$

There finally remains the case where the foci of (35) are fixed, and permutating z and w, this falls under the cases treated in § 9 and 10.

8. **Cases where the conics in the z-plane have a focus variable with t which is a branch point of the right-hand member in (34) and (36).** In any such case, the right-hand member in (34) and (36) cannot be of the form $\varphi(z)\psi(t)$ since the branch points of $\varphi(z)$ are fixed. By the remark at the end of § 4, w is therefore an algebraic function of z. Now the branch points of the left-hand member in (34) are those of $w(z)$ and consequently fixed, while the right-hand member has at least one variable branch point by hypothesis. This case is, therefore, excluded and there remains the case (36) where the left-hand member cannot be independent of the radical (in which case all the branch points would be fixed). Hence we may solve (36) for the w-radical and obtain

$$\sqrt{(w - w_1)(w - w_2)} = \varphi(z, t)\sqrt{(z - z_1)(z - z_2)}, \tag{54}$$

where $w_1 = w_1(t)$ and $w_2 = w_2(t)$ are the foci in the w-plane, both supposed variable, and similarly z_1 and z_2 the variable foci in the z-plane; $\varphi(z, t)$ is rational in z, w and dw/dz and is, therefore, an algebraic function with fixed branch points (viz., those of w). If only one focus in the w-plane is variable, we divide by the factor $\sqrt{w - w_2}$ which is independent of t, and obtain

$$\sqrt{w - w_1} = \varphi(z, t)\sqrt{(z - z_1)(z - z_2)}, \tag{54a}$$

where the branch points of the algebraic function $\varphi(z, t)$ are now those of w and eventually also the points where $w = w_2$, that is the branch points of φ are again fixed. There are similar modifications when only one focus in the z-plane is variable. In (54), $\varphi(z, t)$ cannot have a fixed zero except $z = \infty$ since for such a zero z_0 we must have $w(z_0) = w_1(t)$ or $w_2(t)$, while both the latter are variable. Moreover, $\varphi(z, t)$ cannot have a variable zero $z(t)$ since, excluding those constant t-values for which $z(t)$ coincides with one of the (fixed) branch points of φ, or for which $w_1(t) = w_2(t)$, we have then a holomorphic expansion

$$\varphi(z, t) = \varphi_0(t)(z - z(t)) + \cdots$$

and (54) gives, assuming for instance that $w = w_1(t)$ for $z = z(t)$,

$$w = w_1(t) + \lambda(t)(z - z(t))^m + \cdots,$$

where $m \geqq 2$, and consequently

$$\frac{dw}{dz} = 0 \qquad \text{for} \qquad z = z(t),$$

which is impossible, the zeros of dw/dz being fixed. Similarly, we see that $\varphi(z, t)$ can have no pole variable with t. Hence $z = z_1(t)$ makes the radical to the left in (54) vanish, so that $w = w_1(t)$ or $w_2(t)$. Consequently w takes only two values for each value of z, and in the same way it is seen that z takes only two values for each value of w, so that the algebraic equation connecting w and z is of the second degree in either (in the case (54a), it is of the first degree in w). By (54), φ^2 is therefore a two-valued algebraic function of z and satisfies an equation of the form

$$\alpha(z, t)\varphi^4 + 2\beta(z, t)\varphi^2 + \gamma(z, t) = 0,$$

where α, β and γ are polynomials in z, and since φ has no zero except $z = \infty$ and no pole variable with t, it follows that

$$\gamma(z, t) = \gamma(t), \qquad \alpha(z, t) = \alpha(z)\alpha_1(t),$$

and we find

$$\varphi^2 = \frac{-\beta(z, t) \pm \sqrt{\beta(z, t)^2 - \alpha(z)\alpha_1(t)\gamma(t)}}{\alpha(z)\alpha_1(t)}.$$

But φ^2 has no branch points variable with t, and consequently

$$\beta(z, t)^2 = \alpha(z)\alpha_1(t)\gamma(t) + \delta(z)\delta_1(t)$$

where $\delta(z)$ is a polynomial. If $\delta(z)$ equals a constant times $\alpha(z)$, it is seen at once that φ^2 takes the form

$$\varphi^2 = \frac{1}{\sqrt{\alpha(z)}}\chi(t).$$

Suppose now that $\delta(z)$ and $\alpha(z)$ are linearly independent; any zero of $\beta(z, t)$ which is variable with t must satisfy the equations

$$\alpha(z) \cdot \alpha_1\gamma + \delta(z) \cdot \delta_1 = 0,$$
$$\alpha'(z) \cdot \alpha_1\gamma + \delta'(z) \cdot \delta_1 = 0,$$

and consequently $\alpha(z)\delta'(z) - \alpha'(z)\delta(z) = 0$, so that $\delta(z) = \text{const.}\ \alpha(z)$ contrary to the hypothesis. Consequently the zeros of $\beta(z, t)$ are fixed, so that $\beta(z, t) = \beta(z)\beta_1(t)$, and the identity

$$\beta(z)^2\beta_1(t)^2 = \alpha(z)\alpha_1(t)\gamma_1(t) + \delta(z)\delta_1(t)$$

gives

$$0 = \alpha(z) \cdot \frac{d}{dt}\frac{\alpha_1\gamma}{\beta_1^2} + \delta(z) \cdot \frac{d}{dt}\frac{\delta_1}{\beta_1^2};$$

the linear independence of $\alpha(z)$ and $\delta(z)$ requires that $\beta_1(t)^2 = \text{const.}\ \alpha_1(t)\gamma(t)$, $\delta_1(t) = \text{const.}\ \alpha_1(t)\gamma(t)$, and φ^2 takes the form

$$\varphi^2 = \sqrt{\frac{\gamma(t)}{\alpha_1(t)}} \cdot \psi(z).$$

It is thus shown that (54) may always be written in the form

$$(w - w_1)(w - w_2) = \psi(z)\chi(t)(z - z_1)(z - z_2) \tag{55}$$

(and similarly 54*a*). Now differentiate in respect to t:

$$-(w_1 + w_2)'\cdot w + (w_1w_2)' = \psi(z)[\chi'z^2 - (\chi(z_1 + z_2))'z + (\chi z_1z_2)'],$$

and if $(w_1 + w_2)'$ and $(w_1w_2)'$ are linearly independent, we may eliminate $\psi(z)$ by writing this equation for two different values of t and dividing member by member, thus obtaining w as a rational function of z. In this case, writing (55) in the form

$$(z - z_1)(z - z_2) = \frac{1}{\psi(z)} \cdot \frac{1}{\chi(t)}(w - w_1)(w - w_2),$$

and proceeding in the same way, we find that when $(z_1 + z_2)'$ and $(z_1z_2)'$ are linearly independent, z is also a rational function of w, that is, the algebraic equation connecting w and z is of the first degree in either. The degenerate cases like (54*a*) lead to the same conclusions, and permuting eventually w and z, we see that there are only two cases to be considered: First, the relation between w and z is

$$w = \frac{\alpha z + \beta}{\gamma z + \delta},$$

($\alpha, \beta, \gamma, \delta$ constants) and second, a linear relation exists between $(z_1 + z_2)'$ and $(z_1z_2)'$. In the first case, suppose $\gamma \neq 0$; replacing w and z by expressions of the form $lw + m$, $nz + p$, the relation between them may be reduced to the form

$$w = \frac{1}{z} \tag{56}$$

and our conics in the z-plane, which are not circles or straight lines by hypothesis, are transformed into curves of the fourth degree in the w-plane.*

$$x = \frac{u}{u^2 + v^2}, \qquad y = \frac{-v}{u^2 + v^2}.$$

There remains the assumption $\gamma = 0$, or

$$w = mz + n, \tag{57}$$

which evidently transforms the five-parameter family of all real conics is the z-plane into the corresponding family in the w-plane.

In the second case, we have either $(z_1 + z_2)' = 0$, so that $z_1 + z_2 = c_1$, or $(z_1 z_2)' = c_2(z_1 + z_2)'$, so that $(z_1 - c_2)(z_2 - c_2) = c_3$. By suitable linear transformations, these relations are reduced to the forms $z_1 + z_2 = 0$ and $z_1 z_2 = 1$ respectively.

First, let

$$z_1(t) z_2(t) = 1,$$

and denote by

$$G(z, w) = 0$$

the algebraic equation connecting z and w, and which is of the second degree in z. Since $z_1(t)$ and $z_2(t)$ are the two roots of the equation

$$G(z, w_1(t)) = 0,$$

and $w_1(t)$ is variable, it follows that for any w, the two roots of $G(z, w) = 0$ are z and $1/z$. The two equations

$$G(z, w) = 0 \qquad \text{and} \qquad G\left(\frac{1}{z}, w\right) = 0$$

being identical, it follows that both transform the given family of focal conics in the z-plane into the given family of conics in the w-plane, and consequently the relation

$$Z = \frac{1}{z}$$

transforms our family of focal conics in the z-plane into itself, which conclusion we have recognized as impossible in connection with (56).

Now let

$$z_1(t) + z_2(t) = 0,$$

so that $d = 0$, and by (12), $b = 0$. By the same argument as before, it follows that the two roots of $G(z, w) = 0$ are z and $-z$. Moving the origin of the w-plane, we may assume that one of the branch points of the algebraic function $z(w)$ is $w = 0$. When w makes a circuit about the origin, z changes into $-z$. Now $e(t)$ is not identically zero, since then $z_1 = z_2 = 0$, and if $E(t)$ is not identically zero, we may choose an $\epsilon > 0$ and an interval for t such that $|e(t)| > \epsilon$, $|E(t)| > \epsilon$. Making the circuit about $w = 0$ sufficiently small, it will not, therefore, enclose any branch point of $\sqrt{Cw^2 + 2Dw + E}$ or of $\sqrt{cz^2 + e}$, and the result of continuing (36) analytically along this circuit is then, remembering that $b = d = 0$,

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* This is most readily seen by using rectangular coördinates and transforming an equation of the second degree in x and y by the formulas

$$\frac{A\sqrt{Cw^2 + 2Dw + E} + Cw + D}{(A'w + B')\sqrt{Cw^2 + 2Dw + E} + \frac{1}{2}(C'w^2 + 2D'w + E')}\,\frac{dw}{d(-z)}$$
$$= \frac{a\sqrt{cz^2 + e} - cz}{-a'z\sqrt{cz^2 + e} + \frac{1}{2}(c'z^2 + e')}.$$

Comparing this to (36) we find

$$\frac{a\sqrt{cz^2 + e} + cz}{a'z\sqrt{cz^2 + e} + \frac{1}{2}(c'z^2 + e')} = \frac{-a\sqrt{cz^2 + e} + cz}{-a'z\sqrt{cz^2 + e} + \frac{1}{2}(c'z^2 + e')},$$

which is exactly the condition, investigated in § 7, that the right-hand member in (36) shall have no variable branch point. This case is therefore to be discarded, and we must assume that $E(t)$ vanishes identically. Then $w_2 = 0$, $G(z, w)$ is of the first degree in w, and $G(z, w) = G(-z, w)$ identically; since $w = 0$ is a branch point of $z(w)$, it follows that $G(z, w) = 0$ takes the form

$$w = \frac{\alpha z^2}{\beta z^2 + \gamma}.$$

According as $\beta \neq 0$ or $\beta = 0$, this reduces, upon multiplying z and w by suitable constants, to

$$w = \frac{z^2}{z^2 + 1} \quad \text{or} \quad w = z^2,$$

and the conics

$$\bar{z} = az + \sqrt{cz^2 + e}$$

and

$$\bar{w} = Aw + B + \sqrt{w(Cw + 2D)}$$

correspond.

In the first case,

$$\bar{w} = A\frac{z^2}{z^2 + 1} + B + \frac{z\sqrt{(C + 2D)z^2 + 2D}}{z^2 + 1},$$

and since we also have

$$\bar{w} = \frac{\bar{z}^2}{\bar{z}^2 + 1},$$

it follows that

$$\frac{(A+B)z^2+B+z\sqrt{(C+2D)z^2+2D}}{z^2+1} = \frac{(a^2+c)z^2+e+2az\sqrt{cz^2+e}}{(a^2+c)z^2+e+1+2az\sqrt{cz^2+e}}. \tag{58}$$

When $a = 0$, the conics in the z-plane are the equilateral hyperbolas (44) which are mapped by $Z = z^2$ on the straight lines (45), and

$$w = \frac{Z}{Z+1}$$

maps these on straight lines or circles in the w-plane, instead of on focal conics. Hence $a \neq 0$; making $z = \pm i$ in the equation above, it follows that $-A \pm i\sqrt{-C} = 0$ or $A = C = 0$ (since we cannot have

$$e + 1 - a^2 - c \pm 2ai\sqrt{e - c} = 0,$$

which would imply $c = e$ or $z_1{}^2 = -1$).

Therefore, the left-hand member in (58) has no branch points, while the right-hand member has the branch points $\pm z_1$, since $a \neq 0$, so that this case is impossible.

There finally remains the case

$$w = z^2, \tag{59}$$

transforming the family of conics (12) with $z_2 = -z_1$ or

$$\bar z = k\frac{\bar z_1}{z_1}z + \frac{\bar z_1}{z}\sqrt{(k^2-1)(z^2-z_1{}^2)} \tag{60}$$

and having three parameters k, r, θ, into the conics (with $w_2 = 0$ on account of $E = 0$)

$$\bar w = K\frac{\bar w_1}{w_1}w + \frac{1-K}{2}\bar w_1 + \frac{\bar w_1}{w_1}\sqrt{(K^2-1)w(w-w_1)}; \tag{61}$$

by (54a), it follows that $w = w_1$ for $z = z_1$, or by (59) $w_1 = z_1{}^2$, and writing the conditions that (61) shall result from (60) by means of (59) in the same manner as in the preceding case, we find that these conditions reduce to $K = 2k^2 - 1$. Thus the elements of the conics (61) expressed in terms of those of (60) are

$$w_1 = z_1{}^2, \qquad w_2 = 0, \qquad K = 2k^2 - 1. \tag{62}$$

In the particular case $k = 0$, we obtain (44) and (45).

9. **Cases where the conics in the z-plane are ellipses or hyperbolas with both foci fixed.** Placing the foci at $z = \pm 1$, the equation of our conics is (11), and the right-hand member in (34) and (36) becomes

$$\frac{k\sqrt{(k^2-1)(z^2-1)}+(k^2-1)z}{k'z\sqrt{(k^2-1)(z^2-1)}+kk'(z^2-1)}=\frac{\sqrt{k^2-1}}{k'}\cdot\frac{1}{\sqrt{z^2-1}}.$$

Since this is a product of a function of z by a function of t, the case mentioned in the last sentence of § 7 cannot occur here. According to (25) we now write

$$\frac{dZ}{dz}=\frac{1}{\sqrt{z^2-1}}.$$

whence

$$Z=\log(z+\sqrt{z^2-1}),\qquad z=\tfrac{1}{2}(e^Z+e^{-Z}),\qquad \sqrt{z^2-1}=\tfrac{1}{2}(e^Z-e^{-Z}),\tag{63}$$

and introducing this in (11), we find

$$e^{\bar{Z}}+e^{-\bar{Z}}=(k+\sqrt{k^2-1})e^Z+(k-\sqrt{k^2-1})e^{-Z},$$

and this quadratic in $e^{\bar{Z}}$ evidently has the two roots

$$e^{\bar{Z}}=(k+\sqrt{k^2-1})e^Z,\qquad (k-\sqrt{k^2-1})e^{-Z}.$$

Hence

$$\bar{Z}-Z=\log(k+\sqrt{k^2-1}),\tag{64}$$

which represents a real straight line, viz., a parallel to the real axis, when and only when $1>k>-1$, that is, when the conic in the z-plane is a hyperbola, or

$$\bar{Z}+Z=\log(k-\sqrt{k^2-1})=-\log(k+\sqrt{k^2-1})\tag{65}$$

representing a real straight line, parallel to the imaginary axis, when and only when $k>1$, the conic being then an ellipse. Changing Z into iZ in (63), we find

$$z=\cos Z,\tag{66}$$

transforming the ellipses and hyperbolas into parallels to the real and imaginary axes in the Z-plane respectively.*

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* Writing $Z=\log w$ in (63), so that

$$z=\frac{1}{2}\left(w+\frac{1}{w}\right)$$

it follows that the ellipses and hyperbolas with foci at $z=\pm 1$ are transformed into the circles with center at the origin and straight lines through the origin respectively in the w-plane. This is readily verified from first principles, thus: We have

$$z\pm 1=\frac{1}{2}\left(\sqrt{w}\pm\frac{1}{\sqrt{w}}\right)^2,$$

and for $w=\rho e^{\vartheta i}$,

$$\begin{aligned}|z\pm 1|&=\tfrac{1}{2}\,|\rho^{\frac{1}{2}}e^{\frac{1}{2}\vartheta i}\pm\rho^{-\frac{1}{2}}e^{-\frac{1}{2}\vartheta i}|^2\\&=\tfrac{1}{2}(\rho^{\frac{1}{2}}e^{\frac{1}{2}\vartheta i}\pm\rho^{-\frac{1}{2}}e^{-\frac{1}{2}\vartheta i})(\rho^{\frac{1}{2}}e^{-\frac{1}{2}\vartheta i}\pm\rho^{-\frac{1}{2}}e^{\frac{1}{2}\vartheta i})\\&=\frac{1}{2}\left(\rho+\frac{1}{\rho}\right)\pm\cos\vartheta,\end{aligned}$$

10. Cases where the conics in the z-plane are parabolas with fixed focus. Placing the origin at the focus, the equation of our parabolas is according to (14):

$$\bar{z} = e^{-2\theta i} z - p e^{-\theta i} + 2 i e^{-\frac{3}{2}\theta i} \sqrt{pz},$$

which may also be written

$$e^{\frac{1}{2}\theta i} \sqrt{\bar{z}} - e^{-\frac{1}{2}\theta i} \sqrt{z} = i \sqrt{p}, \tag{67}$$

and the right-hand member of (34) and (36) is, by (21)

$$\frac{\dfrac{\partial \bar{z}}{\partial z}}{\dfrac{\partial \bar{z}}{\partial t}} = \frac{\dfrac{\partial \sqrt{\bar{z}}}{\partial z}}{\dfrac{\partial \sqrt{\bar{z}}}{\partial t}},$$

or using (67),

$$\frac{1}{- 2 i \theta' z + e^{\frac{1}{2}\theta i}(\theta' p^{\frac{1}{2}} + i p^{-\frac{1}{2}} p') \sqrt{z}}. \tag{68}$$

Consider first the cases where this expression is the product of a function of z by a function of t, namely, $\theta' = 0$ and

$$e^{\frac{1}{2}\theta i}(\theta' p^{\frac{1}{2}} + i p^{-\frac{1}{2}} p') = - 2 i \sqrt{c_1} \theta', \tag{69}$$

where c_1 is a constant and $\theta' \neq 0$. In the first case, $\theta =$ const., the axis of the parabolas (67) is fixed and may be taken as the real axis, so that $\theta = 0$. Now (25) and (68) give

$$\frac{dZ}{dz} = \frac{1}{2 \sqrt{z}},$$

and we have the result that

$$Z = \sqrt{z} \tag{70}$$

transforms the one-parameter family of parabolas with focus at the origin and the real axis as axis

$$\sqrt{\bar{z}} - \sqrt{z} = i \sqrt{p} \tag{71}$$

into the straight lines parallel to the real axis

$$\bar{Z} - Z = i \sqrt{p}. \tag{72}$$

In the second case, the differential equation (69) is readily integrated and gives

$$\sqrt{p} = - i \sqrt{c_1} e^{-\frac{1}{2}\theta i} + c_2 e^{\frac{1}{2}\theta i}$$

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whence, for ρ constant, the ellipse

$$| z + 1 | + | z - 1 | = \rho + \frac{1}{\rho},$$

and for ϑ constant, the hyperbola

$$| z + 1 | - | z - 1 | = 2 \cos \vartheta.$$

c_2 being the integration constant, and since $\sqrt{p}$ is real, we must have $c_2 = i\sqrt{\bar{c}_1}$ or

$$\sqrt{p} = -i\sqrt{c_1}e^{-\frac{1}{2}\theta i} + i\sqrt{\bar{c}_1}e^{\frac{1}{2}\theta i}.$$

Now (25), (68) and (69) give

$$\frac{dZ}{dz} = \frac{1}{2}\frac{1}{z + \sqrt{c_1}\sqrt{z}};$$

multiplying z by a constant we may assume that $\sqrt{c_1}$ is real, and we have the result that

$$Z = \log(\sqrt{z} + \sqrt{c_1}) \tag{73}$$

transforms the one parameter family of parabolas with focus at the origin and passing through the fixed point $z = c_1$

$$e^{\frac{1}{2}\theta i}(\sqrt{\bar{z}} + \sqrt{\bar{c}_1}) - e^{-\frac{1}{2}\theta i}(\sqrt{z} + \sqrt{c_1}) = 0 \tag{74}$$

into the straight lines

$$\bar{Z} - Z + \theta i = 0 \tag{75}$$

parallel to the real axis.

When (68) is not the product of a function of z by a function of t, we know from § 4 that w is an algebraic function of z. Consider first equation (34) which becomes by (68)

$$\frac{1}{Lw^2 + Mw + N}\frac{dw}{dz} = \frac{1}{-2i\theta' z + e^{\frac{1}{2}\theta i}(\theta' p^{\frac{1}{2}} + ip^{-\frac{1}{2}}p')\sqrt{z}}; \tag{76}$$

giving a constant value to t, making a linear transformation on w according to (32), and remembering that w must be algebraic, we obtain

$$\frac{1}{w}\frac{dw}{dz} = \frac{1}{2}\frac{m}{z + c_1\sqrt{z}},$$

whence

$$w = (\sqrt{z} + c_1)^m$$

with m real and rational. Substituting in (76), we see as in the case of equation (51) that $m = 1$, so that we may make $c_1 = 0$ by moving the origin in the w-plane. From (67) we now immediately have the result that

$$w = \sqrt{z} \tag{77}$$

transforms the two-parameter family (67) of all parabolas with focus at the origin into the two-parameter family

$$e^{\frac{1}{2}\theta i}\bar{w} - e^{-\frac{1}{2}\theta i}w = i\sqrt{p} \tag{78}$$

of all straight lines in the w-plane.

Next we take up equation (36) with at least one variable focus in the w-plane. This focus cannot be a branch point of the left-hand member of (36), since the right-hand member, or (68), has no variable branch points, and we have therefore in the w-plane the case treated in § 7 for the z-plane, so that, according to (42), equation (36) becomes

$$\frac{2Cw}{C'w^2 + E'}\frac{dw}{dz} = \frac{1}{-2i\theta'z + e^{\frac{1}{2}\theta i}(\theta'p^{\frac{1}{2}} + ip^{-\frac{1}{2}}p')\sqrt{z}}. \tag{79}$$

Giving t a constant value, and multiplying w and z by suitable constants, we obtain a differential equation which gives upon integration

$$\sqrt{z} + c_2 = (w^2 + c_1)^m$$

and substituting this in (79), we obtain $m = 1$ as before. Replacing $c_1 - c_2$ by $-\sqrt{c_1}$, we have the result that

$$w^2 = \sqrt{z} + \sqrt{c_1}, \tag{80}$$

where we may assume $\sqrt{c_1}$, real, transforms the two-parameter family (67) of all parabolas with focus at the origin into the two-parameter family

$$e^{\frac{1}{2}\theta i}\bar{w}^2 - e^{-\frac{1}{2}\theta i}w^2 = i(\sqrt{p} + 2\sqrt{c_1}\sin\tfrac{1}{2}\theta) \tag{81}$$

of all equilateral hyperbolas with the origin as center in the w-plane.

Finally, consider equation (36) with fixed foci in the w-plane. By § 9, confocal ellipses or hyperbolas in the w-plane would give (68) the form of a product of a function of z by a function of t, which case is excluded by hypothesis. Hence the conics in the w-plane are parabolas, and we may take the common focus as origin. The parameters of these parabolas being P and ϑ, (68) shows that (36) takes the form

$$\begin{aligned}\frac{1}{-2i\vartheta'w + e^{\frac{1}{2}\vartheta i}(\vartheta'P^{\frac{1}{2}} + iP^{-\frac{1}{2}}P')\sqrt{w}}\frac{dw}{dz}\\ = \frac{1}{-2i\theta'z + e^{\frac{1}{2}\theta i}(\theta'p^{\frac{1}{2}} + ip^{-\frac{1}{2}}p')\sqrt{z}}.\end{aligned} \tag{82}$$

Giving t a constant value, multiplying z and w by suitable constants and integrating the resulting differential equation, we find

$$\sqrt{w} + \sqrt{c_2} = (\sqrt{z} + \sqrt{c_1})^m$$

and substituting in (82), it is seen as before that $m = 1$ (and $c_2 = 0$, c_1 real), whence the result that

$$\sqrt{w} = \sqrt{z} + \sqrt{c_1} \tag{83}$$

transforms the two-parameter family (67) of all parabolas with focus at the origin into itself with a changed parameter p:

$$e^{\frac{1}{2}\theta i}\sqrt{\bar{w}} - e^{-\frac{1}{2}\theta i}\sqrt{w} = i(\sqrt{p} + 2\sqrt{c_1}\sin\tfrac{1}{2}\theta). \tag{84}$$

11. **Summary of results.** Replacing z and w by $az + b$ and $\alpha w + \beta$ respectively, or by $(az + b)/(cz + d)$ and $(\alpha w + \beta)/(\gamma w + \delta)$ when the conics involved are circles, and interchanging z and w when necessary, we may reduce our conformal maps to the following types, arranged according to the number of parameters in the families of conics:

Five parameters (one type):

I. $$w = z;$$

any conic in the z-plane is transformed into the same conic in the w-plane.

Three parameters (two types):

II. $$w = 1/z;$$

any circle or straight line is transformed into a circle or straight line.

III. $$w = z^2;$$

denoting the parameters by k, r and θ, any ellipse ($k > 1$) or hyperbola ($1 > k > -1$)

$$\bar{z} = k\frac{\bar{z}_1}{z_1}z + \frac{\bar{z}_1}{z_1}\sqrt{(k^2 - 1)(z^2 - z_1^2)}$$

with center at the origin and foci at

$$z_1 = re^{\theta i}, \qquad z_2 = -z_1$$

is transformed into the ellipse ($k > 1$) or hyperbola ($1 > k > -1$)

$$\bar{w} = K\frac{\bar{w}_1}{w_1}w + \frac{1 - K}{2}\bar{w}_1 + \frac{\bar{w}_1}{w_1}\sqrt{(K^2 - 1)w(w - w_1)}$$

with $K = 2k^2 - 1$ and foci at

$$w_1 = z_1^2 = r^2e^{2\theta i}, \qquad w_2 = 0.$$

In the special case $k = 0$, the equilateral hyperbolas

$$e^{-2\theta i}z^2 + e^{2\theta i}\bar{z}^2 = r^2$$

with center at the origin go into the straight lines

$$e^{-2\theta i}w + e^{2\theta i}\bar{w} = r^2.$$

Two parameters, denoted by r, θ or p, θ (four types)

IV. $$w^2 = z^2 - 1;$$

any equilateral hyperbola

$$e^{-2\theta i}z^2 + e^{2\theta i}\bar{z}^2 = r^2$$

with center at the origin and foci at $z_1 = re^{\theta i}$, $z_2 = -z_1$ is transformed into the equilateral hyperbola

$$e^{-2\theta i}w^2 + e^{2\theta i}\bar{w}^2 = r^2 - 2\cos 2\theta$$

with center at the origin and foci at

$$w_1 = \sqrt{r^2 - 2\cos 2\theta}\, e^{\theta i}, \qquad w_2 = -w_1.$$

In the special case $r^2 = 2\cos 2\theta$, we have a pair of perpendicular straight lines through the origin in the w-plane.

V. $$w = z^2;$$

any straight line

$$e^{-\frac{1}{2}\theta i}\bar{z} - e^{-\frac{1}{2}\theta i}z = i\sqrt{p}$$

is transformed into the parabola

$$e^{-\frac{1}{2}\theta i}\sqrt{\bar{w}} - e^{-\frac{1}{2}\theta i}\sqrt{w} = i\sqrt{p}$$

with focus at the origin and at a distance $\frac{1}{2}p$ from the directrix, θ being the angle between the axis of the parabola and the real axis in the w-plane.

VI. $$w = (z^2 - c_1)^2, \qquad \text{where } c_1 = 0 \text{ or } 1;$$

any equilateral hyperbola

$$e^{\frac{1}{2}\theta i}\bar{z}^2 - e^{-\frac{1}{2}\theta i}z^2 = i(\sqrt{p} + 2c_1 \sin \tfrac{1}{2}\theta)$$

with center at the origin and foci at

$$z_1 = e^{\frac{1}{4}(\theta - \pi)i}(\sqrt{p} + 2c_1 \sin \tfrac{1}{2}\theta)^{\frac{1}{2}}, \qquad z_2 = -z_1$$

is transformed into the parabola

$$e^{\frac{1}{2}\theta i}\sqrt{\bar{w}} - e^{-\frac{1}{2}\theta i}\sqrt{w} = i\sqrt{p}$$

with focus at the origin.

VII. $$w = (\sqrt{z} - 1)^2;$$

any parabola

$$e^{\frac{1}{2}\theta i}\sqrt{\bar{z}} - e^{-\frac{1}{2}\theta i}\sqrt{z} = i(\sqrt{p} + 2\sin \tfrac{1}{2}\theta)$$

with focus at the origin is transformed into the parabola

$$e^{\frac{1}{2}\theta i}\sqrt{\bar{w}} - e^{-\frac{1}{2}\theta i}\sqrt{w} = i\sqrt{p}$$

with focus at the origin.

One parameter, denoted by r, θ or p (nine principal types, the linear composition of each of these with itself or with another giving a total of forty-five types). There are tabulated below the nine functions which map the one-parameter families of conics specified on the parallels to the real axis in the Z-plane. When $\varphi_1(z)$ and $\varphi_2(z)$ are any two of these functions (identical or different), all maps transforming one-parameter families of conics into others are found by making

$$\varphi_2(w) = A\varphi_1(z) + B,$$

where A and B are constants and A *real*, and then making the linear transformations on z and w indicated at the beginning of this summary.

VIII. $$Z = z;$$

all parallels to the real axis (special case of I).

IX. $$Z = \log z;$$

all straight lines through $z = 0$ (the circles with center at $z = 0$ go into parallels to the imaginary Z-axis).

X. $$Z = i \log z;$$

all circles with center at $z = 0$ (the straight lines through $z = 0$ go into parallels to the imaginary Z-axis).

XI. $$Z = z^2;$$

all equilateral hyperbolas $z^2 + \bar{z}^2 = r^2$ with center at the origin and foci on the real axis (special case of III).

XII. $$Z = \log (z^2 - 1);$$

all equilateral hyperbolas

$$e^{-2\theta i}z^2 + e^{2\theta i}\bar{z}^2 = 2 \cos 2\theta$$

with center at the origin and foci $\pm \sqrt{2 \cos 2\theta}\, e^{\theta i}$ (which are the end points of the diameter with angle of slope θ of the lemniscate with foci at $+1$ and -1). These hyperbolas all pass through the foci of the lemniscate.

XIII. $$Z = \log (z + \sqrt{z^2 - 1}) \quad \text{or} \quad z = \tfrac{1}{2}(e^Z + e^{-Z});$$

all hyperbolas with foci at $+1$ and -1 (all ellipses with these foci go into parallels to the imaginary Z-axis).

XIV. $$Z = \frac{1}{i} \log (z + \sqrt{z^2 - 1}) \quad \text{or} \quad z = \cos Z;$$

all ellipses with foci at $+1$ and -1 (all ellipses with these foci go into parallels to the imaginary Z-axis).

XV.
$$Z = \sqrt{z};$$
all parabolas $\sqrt{\bar{z}} - \sqrt{z} = i\sqrt{p}$ with focus at the origin and the real axis as axis (special case of V).

XVI.
$$Z = \log(\sqrt{z} - 1);$$
all parabolas
$$e^{\frac{1}{2}\theta i}\sqrt{\bar{z}} - e^{-\frac{1}{2}\theta i}\sqrt{z} = 2i \sin \tfrac{1}{2}\theta$$
with focus at the origin and passing through the point $z = 1$. As examples of the possible combinations, we may mention the following: It was shown in § 5 that
$$w = z^k$$
transforms the two families of straight lines through the origin and circles with center at the origin into themselves when k is real and permutes the two families when k is purely imaginary, and $w = z^k$ is obtained by composing VIII with itself:
$$\log w = k \log z$$
when k is real, but VIII with IX
$$\log w = \frac{k}{i} \cdot i \log z$$
when k is purely imaginary.

The map
$$w = \frac{c}{2}(z + \sqrt{z^2 - 1})^m + \frac{1}{2c}(z - \sqrt{z^2 - 1})^m$$
where m is real, transforms the families of all ellipses and all hyperbolas with foci at $+1$ and -1 into themselves, since it is obtained by composing XIII or XIV with itself:
$$\log(w + \sqrt{w^2 - 1}) = m \log(z + \sqrt{z^2 - 1}) + \log c,$$
or
$$\frac{1}{i}\log(w + \sqrt{w^2 - 1}) = m \cdot \frac{1}{i}\log(z + \sqrt{z^2 - 1}) + \frac{1}{i}\log c.$$